# Differential Geometry I: Curves and surfaces

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**1.1** Recall that the cross product of two vectors in $\mathbb{R}^3$, expressed with respect to a positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ as

$$x = x_1e_1 + x_2e_2 + x_3e_3, \quad y = y_1e_1 + y_2e_2 + y_3e_3$$

is the vector

$$x \times y \doteq (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3.$$

Prove that, for any $a, b \in \mathbb{R}^3$, the cross product $a \times b$ is uniquely determined by the following geometric conditions:

**(a)** $(a \times b) \perp a$ and $(a \times b) \perp b$,

**(b)** $\|a \times b\| = \text{Area}(P(a, b))$, where $P(a, b)$ is the parallelogram spanned by the vectors $a$ and $b$,

**(c)** If $a$ and $b$ are linearly independent, then $\{a, b, a \times b\}$ is a positively oriented basis of $\mathbb{R}^3$.

**1.2** Let $\|\cdot\|$ be a norm on the real vector space $V$. Assume that $\|\cdot\|$ satisfies the *parallelogram law*: For any $x, y \in \mathbb{V}$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Prove that $\|\cdot\|$ is a Euclidean norm, i.e. there exists an inner product $\langle \cdot, \cdot \rangle$ on $V$ such that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in V$.

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**The isoperimetric inequality in the plane.** The aim of Exercises $1.3 - -1.9$ is to establish the isoperimetric inequality in the plane. Let us consider a bounded domain $D \subset \mathbb{R}^2$. Its boundary $\partial D$ is the union of one or more curves, and the perimeter of $D$ is defined as the total length of $\partial D$ (which could be infinite). The isoperimetric quotient of $D$ is defined by

$$\text{Isp}(D) = \frac{(\text{Length}(\partial D))^2}{\text{Area}(D)}.$$

The isoperimetric inequality states that for every bounded domain $D \subset \mathbb{R}^2$, we have

$$\text{Isp}(D) \geqslant \text{Isp}(\mathbb{B}^2),$$

where $\mathbb{B}^2 = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ is the unit disk. Equality holds if and only if $D$ is a disk (of any radius).

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**1.3** Let $D \subset \mathbb{R}^2$ be a bounded domain.

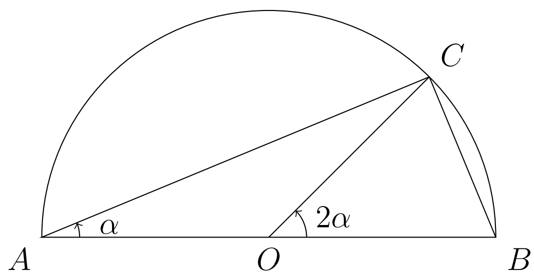
**(a)** Prove that the isoperimetric quotient is invariant under similarity transformations (*Recall that a similarity transformation of a Euclidean space is a bijection which preserves the ratios of distances; it can always be expressed as the composition of a homothetic map* $x \to \lambda x$ *for a* $\lambda > 0$ *and an isometry*).

**(b)** Compute the isoperimetric quotient of a square, of an equilateral triangle, and of a disk.

**1.4** For this exercise, you will have to recall some basic facts from Euclidean geometry. (a) Let $C$ be a point on the circle with diameter $[A, B]$ (see the figure below). Prove that

$$\widehat{COB} = 2\widehat{CAB}.$$

(b) Deduce the semi-circle theorem of Thales: A triangle $ABC$ is right-angled at $C$ (i.e. $\widehat{ACB} = \frac{\pi}{2}$) if and only if $C$ lies on the circle with diameter $[A, B]$.




**1.5** Prove that among all triangles $ABC$ such that $d(A, C) = x$ and $d(B, C) = y$ with $x, y$ fixed, the one with maximal area is the right triangle with right angle at $C$.

**1.6** Recall that a domain $D \subset \mathbb{R}^2$ is called *convex* if, for any two points $A, B \in D$, the line segment $[A, B]$ is contained inside $D$. Prove that if $D$ is not convex, then $D$ does not minimize the isoperimetric quotient, i.e. you can construct a new domain $D'$ such that $\text{Isp}(D) > \text{Isp}(D')$ (*Hint: Starting from a segment with endpoints on* $\partial D$ *but not lying entirely in* $D$*, show that you can construct a domain* $D'$ *with Length*$(\partial D') \leqslant$ *Length*$(\partial D)$ *but Area*$(D') >$ *Area*$(D)$).

**1.7** Suppose $D$ is an isoperimetrically optimal domain (in particular, $D$ is convex) and let $\Gamma = \partial D$ (since $D$ is convex, $\Gamma$ is a single connected curve). Suppose that $A, B \in \Gamma$ are two points with the property that they divide $\Gamma$ into two arcs of equal length. Show that the line segment $[A, B]$ divides $D$ into two regions of equal area.

**1.8** Suppose $D$ is an isoperimetrically optimal domain and let the points $A, B \in \partial D$ have the property that they divide $\partial D$ into two arcs of equal length. Show that for every point $P \in \partial D$ distinct from $A, B$, we have $\widehat{APB} = \frac{\pi}{2}$.

*Hint: Assume, for the sake of contradiction, that there exists a* $P \in \partial D$ *with* $\widehat{APB} \neq \frac{\pi}{2}$*. Starting from the triangle* $APB$ *and using Exercise 1.5, show that you can create a domain* $D'$ *with Length*$(\partial D') =$ *Length*$(\partial D)$ *but Area*$(D') >$ *Area*$(D)$.

**1.9** Using the previous exercises, prove the isoperimetric inequality in the plane:

$$Isp(D) \geqslant 4\pi,$$

with equality if and only if $D$ is a disk.

*Remark.* In proving the above, you can assume that an optimal shape (i.e. a domain $D$ minimizing the value of the isoperimetric ration) exists. Proving the existence of such a minimizer is a harder task and requires the use of more tools from analysis (in particular, a compactness argument analogous to the Arzela–Ascoli theorem).